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EFFECTIVE ELASTIC MODULI OF GRANULAR MEDIA

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The solution of problems concerning the deformation of heterogeneous media is often based on the hypothesis of effective homogeneity [1], which amounts to assuming that the heterogeneous medium can be replaced by a homogeneous continuum having certain effective parameters. The problem consists of determining the effective properties of the heterogeneous medium through the properties of the phases and some of their geometric characteristics. Finding the effective elastic moduli of sandstone oil and gas resevoirs in general and the velocities of longitudinal and transverse waves in particular, determining the relationship between the velocities and the structure of the pore space on the one hand and the properties of the fluid on the other hand – these are very important problems for seismic prospecting.

We will concern ourselves with the simpler situation of an empty (not containing fluid) consolidated granular skeleton. There are several approaches to solving this problem, but until recently the granular character of the skeleton has been accounted for only in solutions based on the Hertz problem concerning the deformation of two spheres at the point of contact under the influence of applied forces [2]. However, the presumption of point contact at the initial moment of loading does not conform to the condition of consolidation of rock and leads to a situation whereby elastic waves in such a model propagate only in the presence of external pressure. A number of other solutions [1] account only for the fraction of the volume corresponding to the pore space. In practice, an equation obtained from statistical analysis of a large number of laboratory measurements is widely used to relate the velocities of elastic waves with a certain characteristic of the structure (mainly porosity), as well as with mean grain size, permeability, etc. Thus, there is a need for new approaches to the calculation of the effective elastic moduli of granular media.

The author of [3] proposed the use of the variational approach to calculate the stress state of an individual main and the effective elastic moduli of an empty granular skeleton. He investigated a longwave approximation, i.e., a situation in which the length of the elastic wave is much greater than the sizes of the grains. This makes it possible to change over to the static equations for an individual grain. A granular body is subjected to a hypothetical unilateral compression

$$e_{11} = e_{22} = e_{21} = e_{13} = e_{23} = 0, \; e_{33} = 1. \qquad (1)$$

It is assumed that the energy associated with the deformation of one grain is minimal with certain restrictions on the character of this deformation, i.e.: the grain as a whole does not undergo displacement or rotation, and the strain tensor at the center of the grain has the form (1). These requirements make it possible to completely determine the unknowns at the points of contact with the loading grain. However, the hypothesis on the character of the elastic strains of an individual grain needs to be more carefully substantiated. In any case, it should be consistent with the asymptotic solution for a continuum.

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We will examine a grain of the volume V. Its surface S consists of areas of contact with adjacent grains having the total area ΔS and the free surface $S - \Delta S$. Let the loads f_i, nontrivial only on ΔS, satisfy certain conditions which are linear with respect to f_i:

$$T^k = \int_{\Delta S} t_i^k(y) f_i(y)\, dS, \qquad (2)$$

i.e., certain restrictions are placed on the character of deformation of the grain. We find the variation of the energy functional

$$E = \frac{1}{2}\int \sigma_{ij} e_{ij} dV.$$

Since $e_{ij} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, and since the strains and stresses are connected by Hooke's law, we write

$$\delta E = -\int \delta u_i U_i dV + \int \frac{\partial F_i}{\partial x_i} dV,$$

where

$$F_i = u_m \left(\lambda \frac{\partial \delta u_k}{\partial x_k} \delta_{mi} + \mu \left(\frac{\partial \delta u_m}{\partial x_i} + \frac{\partial \delta u_i}{\partial x_m}\right)\right);$$

$$U_i = \mu \frac{\partial^2 u_i}{\partial^2 x_k} + (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k}.$$

Since the displacements u_i satisfy the Lamé equations, the first integral is equal to zero. The second integral, in accordance with the divergence theorem, is transformed into a surface integral:

$$\delta E = \int F_i n_i dS = \int u_i(y)\, \delta f_i(y)\, dS. \qquad (3)$$

Given conditions (2), the minimum of functional (3) is obtained if $u_i(y) = -\lambda_k t_i^k(y)$, $y \in \Delta S$ (λ_k are the Lagrange constants). Thus, the energy minimization problem formulated in [3] is reduced to an internal problem of the theory of elasticity with mixed boundary conditions

$$u_i(y) = -\lambda_k t_i^k(y),\ y \in \Delta S, \qquad (4)$$
$$f_i(y) = 0,\ y \in S - \Delta S.$$

We will change over to the case of a solid. In this case, $\Delta S \to S$. The form of the surface of the grain may be arbitrary. We choose a sphere of radius R, which allows us to obtain the result in analytical form. The displacements near the center of the sphere are determined by the first and third terms of the expansion of the surface loads in spherical functions [4]:

$$2\mu \frac{\mathbf{u}}{R} = \frac{r}{R}\mathbf{Y}_1 + \frac{R^2 m}{2(7m+5)} \nabla \operatorname{div}\left(\left(\frac{r}{R}\right)^3 \mathbf{Y}_3\right) - \frac{r}{m+1} \operatorname{div}\left(\frac{r}{R}\mathbf{Y}_1\right). \qquad (5)$$

Here, $\mathbf{Y}_n(\theta, \varphi) = (2n+1)/4\pi R^2 \int \mathbf{f}(\theta, \varphi) P_n(\cos\gamma) dS$; $\cos\gamma = \sin\theta \times \sin\theta' \cos(\varphi - \varphi') + \cos\theta \cos\theta'$, $m = 2(\lambda + \mu)/\lambda$. We find the strain tensor at the center of the sphere through the surface integral over the loads

$$e_{np}(0) = A \int \left(5 y_n \delta_{pi} + 5 y_p \delta_{ni} + 35 \frac{m}{R^2} y_n y_p y_i - B y_i \delta_{np}\right) f_i(y)\, dS \qquad (6)$$
$$(A = (16\pi\mu(7m+5)R^3/3)^{-1},\ B = (7m^2 + 21m + 10)/(m+1))$$

along with the analytical form of the functions $t_i^k(y)$ corresponding to coupling conditions (1). To solve the equilibrium equation with boundary conditions (4) at $\Delta S = S$ and the explicit form of $t_i^k(y)$ determined from (6), we use the solution of the first internal problem for a sphere [4]. Having performed the necessary calculations, we finally obtain on the surface of the sphere displacements which do not conform to the hypothetical unilateral compression:

$$u_i(y) = (1 + D(4 - 7m)) y_i \delta_{3i} + D y_i (2 + (7m - 10) y_3^2/R^2) \qquad (7)$$
$$(D = 7m/(49m^2 - 14m - 20)).$$

Thus, the grain can have a stress state with an energy lower than that present in unilateral compression but with the same form of the strain tensor at the center. As a result, conditions

(1) are not consistent with the asymptotic solution for a continuum.

Considering the above approach to determination of the stress state of a grain, we will make use of the energy method in [1]. This method entails determination of the effective elastic moduli through the equality of the strain energies stored by the heterogeneous and effective homogeneous media. Let a certain volume V of a granular body with a porosity f be subjected to cubic compression. For an effective homogeneous body with the elastic moduli K^* and μ^*, we have the mean stress tensor $\sigma_{ij} = P\delta_{ij}$ and energy $E = VP^2/2K^*$. A stress tensor of this form is the result of averaging over the macrovolume of the granular body. Since the skeleton is empty, we integrate only over the grains. If V_0 is the volume of one grain, then the volume contains $N = V(1 - f)V_0$ grains. Let $\langle\sigma_{ij}^0\rangle$ be the mean stress tensor of a typical grain. Then

$$\sigma_{ij} = \frac{1}{V}\int \sigma_{ij}^0 dV = \frac{1}{V}\langle\sigma_{ij}^0\rangle NV_0.$$

Thus, for the mean stress tensor of the grain

$$\langle\sigma_{ij}^0\rangle = \frac{P}{1-f}\delta_{ij}.$$

We do not pretend that the derivation of this relation was rigorous. It can obviously be considered only an estimate of the mean stress tensor of the grain.

Of all possible states of a grain with this form of mean stress tensor, we will choose the stress state with the minimum energy E_0. The coupling conditions are written very simply, since it is known [5] that

$$\langle\sigma_{ij}^0\rangle = \frac{1}{2V_0}\int (y_i f_j + y_j f_i)\, dS. \tag{8}$$

By virtue of the linearity of the Lamé equations, the loads found on the contacts from the solution of the problem of elasticity theory with mixed boundary conditions (4) can be represented in the form of a series in Lagrange constants. In this case, coupling conditions (8) reduce to a system of linear algebraic equations $B_{nm}^{pk}\lambda_{nm} = -\langle\sigma_{pk}^0\rangle$. For convenience, the Lagrange constants and coupling conditions will be used with a double index. Considering that $E = (1/2)\int u_i f_i dS$ and using boundary conditions (4), we obtain

$$E_0 = -\frac{\lambda_{nm}}{2}\int t_i^{nm}(y) f_i(y)\, dS = -\lambda_{nm}\langle\sigma^0\rangle/2,$$

or

$$E_0 = \langle\sigma_{nm}^0\rangle (B_{pk}^{nm})^{-1}\langle\sigma_{pk}^0\rangle.$$

The energy of the entire volume V is NE_0. Thus, we finally obtain

$$K^* = V_0(1-f)/(B_{ll}^{ll})^{-1}.$$

To determine the second elastic modulus, we proceed as before and pose the problem of a hypothetical pure shear. Then $\sigma_{ij} = P\delta_{i1}\delta_3$, $E = VP^2/2\mu^*$. The effective shear modulus

$$\mu^* = V_0(1-f)/(B_{13}^{13})^{-1},$$

while the velocities of the longitudinal and transverse waves

$$V_P^* = \sqrt{\frac{K^* + 4\mu^*/3}{\rho(1-f)}}, \quad V_S^* = \sqrt{\frac{\mu^*}{\rho(1-f)}}$$

(ρ is the density of the material of the grain). In terms of the method used to construct it, the matrix B depends only on the geometry of the grain and its elastic properties. Here, the elastic constants of the grain appear in the expressions for μ^*/μ, K^*/K, V_P^*/V_P, V_S^*/V_S only in the form of the Poisson's ratio.

To solve the problem with mixed boundary conditions (4) for a grain with a known distribution of contacts, we use the method of integral boundary equations [6]. We represent the displacement vector in the form of the potential of a simple layer

$$u_i(x) = \int \Gamma_{ij}(x, y)\varphi_j(y) dS \tag{9}$$

[$\varphi_j(y)$ is the density of the potential, $\Gamma_{ij}(x, y)$ is a Kelvin–Somilyan matrix]. The potential density is connected with the surface loads by a system of integral boundary equations

$$\varphi_i(y) + \int \Gamma^1_{ij}(y, y')\varphi_j(y')\,dS = f_i(y), \tag{10}$$

which can be solved by the method of successive approximations. To realize this numerically, we subdivide the surface of the grain into elementary areas dS^k, assuming that all of the unknowns are constant and are referred to its center. We will seek the load in the form of a series with constant coefficients a_i^m:

$$f_i(y) = \sum_{m=1}^{M} a_i^m \Omega^m(y),$$

where M is the number of elementary areas at the contacts; $\Omega^m(y)$ is the characteristic function of an area; meanwhile, $\Omega^m(y) = 0$ if $y \notin dS^m$, and $\Omega^m(y) = 1$ if $y \in dS^m$. Using (9) and (10), we represent the displacements on the contacts in the form of a series with the coefficients a_i^m. Mixed boundary conditions (3) reduce to a 3M × 3M system of linear algebraic equations relative to the coefficients a_i^m. This system is solved by Gauss' elimination method.

The effective properties of the granular body cannot be determined without a sufficiently detailed description of its structure. The models used for these purposes should reflect the structural features of the pore space and matrix that are important for the properties being studied. The model which is best suited for these purposes is the model of intersecting spheres [7]. Use of this model makes it possible to change over from a system of contacts distributed on the grain to integral parameters of the structure.

We will examine a system of spheres of radius R_0 which are distributed randomly in space and are in contact with one another at a point (N is the mean number of contacts per sphere and f_0 is the porosity of the system). A whole range of rather similar relations $f_0(N)$ has been obtained by both theoretical and empirical means. We will use the following procedure [8]. Let there be regular packings of spheres of identical radius that differ in porosity and the number of contacts: face-centered cubic $f_0 = 0.2595$, N = 12; body-centered cubic $f_0 = 0.3198$, N = 8; cubic $f_0 = 0.4764$, N = 6; tetrahedral $f_0 = 0.6599$, N = 4. We can depart from a realistic geometric representation and assume that the static parameters of random packings change continuously in accordance with an interpolation curve drawn through the points belonging to the regular packings. We thus find the function f(N) in the range from 4 to 12.

Having fixed the centers of the spheres distributed randomly in space, we increase their radius from R_0 to $R = \sqrt{R_0^2 + r^2}$ (r is the radius of the contact spot). The porosity of such a system of intersecting spheres

$$f = 1 - \frac{1-f_0}{(1-\delta^2)^{3/2}}\left(1 - \frac{N}{4}\left(1 - \sqrt{1-\delta^2}\right)\left(1 + \delta^2 - \sqrt{1-\delta^2}\right)\right),$$

while the specific surface (the surface of the pore space per unit volume)

$$S_V = \frac{3(1-f_0)}{R}\left(1 - \frac{N}{2}\left(1 - \sqrt{1-\delta^2}\right)\right)\Big/(1-\delta^2)^{3/2} \quad (\delta = r/R).$$

Having determined the mean grain size $\langle D\rangle$ as the diameter of a sphere with the same ratio of volume to surface, we introduce a dimensionless parameter in the form of the product of the specific surface and the mean grain size $\eta = S_V\langle D\rangle$. The geometry of such a model is completely determined by two dimensionless parameters — such as f and η — and by the characteristic linear dimension $\langle D\rangle$. Since the wavelength $\ell \gg \langle D\rangle$, it is natural to suggest that the velocities of the elastic waves in such media will be a function of two parameters of the structure of the pore space.

We will use the model of intersecting spheres to calculate the effective elastic moduli. Having assigned the distribution of contacts in a grain in accordance with one of the regular packings described above, we change the radius of the contact spot and, thus, the porosity of the packing. Here solving the problem with mixed boundary conditions (4), we obtain the dependence of the velocities of the longitudinal and transverse waves on porosity for each packing. Figures 1 and 2 show such relations. Here, the Poisson's ratio of the grain was assumed to be equal to 0.25. The numbers next to the curves denote the type of packing. Of course, the elastic energy depends on the distribution of the contacts with respect to direction. Thus, it is more reliable to take the average over an ensemble of grains with the same number of contacts and different distributions with respect to the direction. Moreover, a certain distribution of grains with respect to the number of contacts is realized in random

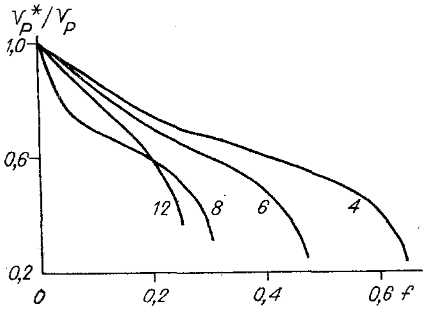

Fig. 1

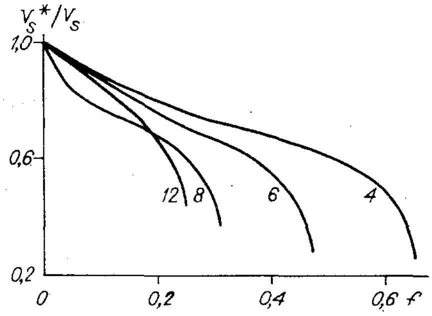

Fig. 2

packings, and it is also necessary to take the average for this distribution. However, it is unlikely that such a procedure will significantly distort the character of the resulting relations, since the main factors affecting the energy of the grains are the number of contacts and their sizes. Thus, we will assume that the curves that were obtained are also valid for random packings with a mean number of contacts per grain (N = 4, 6, 8, 12). The velocities for intermediate values of N — including fractional values — can be found by interpolation. It is more expedient to change over to the parameters f and η. Using the method of multiple regression, we approximate the dependences of V_P^*/V_P and V_S^*/V_S on f and η by the equations

$$V_P^* = V_P(1 - 0.712f - 0.0776\eta),$$
$$V_S^* = V_S(1 - 0.660f - 0.0617\eta).$$

The mean error of the prediction with such an approximation of the curves in Figs. 1 and 2 is 5.6% for the longitudinal waves and 6.2% for the transverse waves.

Thus, the following conclusions can be made. The velocities of longitudinal and transverse waves in consolidated granular media depend on at least two structure parameters of the pore space. Examples of such parameters are porosity and the product of the specific surface and mean grain size. Meanwhile, an increase in the latter leads to a decrease in the wave velocities. The velocities of the longitudinal waves are more sensitive to changes in the structure parameters compared to the transverse waves. This leads to a decrease in the effective Poisson's ratio. The effective elastic parameters depend only on the Poisson's ratio of the material of the matrix.

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